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ON THE PROBLEM OF REINFORCEMENT OF THE HOLE OUTLINE IN A PLATE BY A MOMENTLESS ELASTIC ROD*

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The shape of the hole outline in a plate and the law of stiffness variation of a reinforcing momentless rod are sought from the condition of the minimum total elastic strain energy of the plate and the reinforcement. The area enclosed by the outline of the cutout, and the magnitude of the volume of the reinforcing element are considered given. The boundary conditions of the problem are obtained by varying the functional related to the elastic strain energy in domains with moving boundaries. The solution is constructed in the form of an expansion in terms of a small parameter, for which the parameter of nonsymmetry of the load is taken. The solution obtained in this manner differs substantially from the solution obtained in /1/.

The problem of logical reinforcement of a hole outline in a plane state of stress has been investigated by many authors. The bending stiffness of the reinforcing element concentrated at the hole outline can be neglected and considered a momentless rod. Certain estimates of the error hence induced are presented in /1–4/, for instance. Reinforcement of an outline of given shape (circle, ellipse) by an elastic momentless rod has been investigated in /4–11/. It is shown that a decrease in the stress concentration in the plate can be achieved because of the appropriately selected variable of the section area of the reinforcing rod. Not only the law of variation of the rod section area is sought in the problem of equivalent reinforcement /1/, but also the shape of the cutout for which the same state of stress is conserved outside the reinforcement as exists for a plate without a hole. The solution of this problem exists in a limited band of load relationships, and the volume occupied by the reinforcement exceeds the part removed for the formation of the cutout by several times, as a rule.

1. Formulation of the problem. The plane state of stress of a thin plate with a hole (Fig.1) is considered. At a sufficient distance from the cutout the stresses are

$$\sigma_x = p, \quad \sigma_y = q, \quad \tau_{xy} = 0 \tag{1.1}$$

The hole outline L is reinforced by a momentless elastic rod with variable stiffness under tension $G(s)$. The plate elastic modulus, Poisson's ratio, and thickness are denoted by E, ν, h. The shape of the outline L and the law of variation of the rod stiffness $G(s)$ are sought from the condition of minimum total elastic strain energy of the plate and the reinforcing rod for given area D, domain Ω' bounded by the outline L, and magnitude of the integral H

$$\iint_{\Omega'} dxdy = D, \quad \int_L G(s)\,ds = H \tag{1.2}$$

For a constant elastic modulus of the reinforcement material the last condition means that the volume (or weight) of the reinforcing element is given.

For an unbounded plate we understand the strain energy to be the strain energy of the part of the plate (the domain Ω) included between the hole outline L and some sufficiently remote fixed closed outline L_1 outside which the undisturbed state of stress has the form (1.1).

Let us examine the functional

$$J = \frac{K}{2}\iint_\Omega [u_x^2 + v_y^2 + 2\nu u_x v_y + \gamma_1 (u_y + v_x)^2]\,dxdy + \frac{1}{2}\int_L (-u_s{}^* y_n + v_s{}^* x_n)^2 G(s)\,ds - h\int_{L_1} (p x_n u + q y_n v)\,ds \tag{1.3}$$

$$K = Eh/(1-\nu^2), \quad \gamma_1 = (1-\nu)/2$$

Here u, v are displacement vector components of points of the plate in the Cartesian coordinate system $x0y$, the asterisks denote the appropriate quantities referred to the reinforcement;

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the direction of traversal in the contour integrals is counter-clockwise, s is the arc coordinate measured along the outline from a certain initial point, the subscripts denote differentiation with respect to the appropriate variable $x_n = \cos(n, x)$, $y_n = \cos(n, y)$ are direction cosines of the normal n.

The first two components in the right side of (1.3) are the strain energies of the plate and the reinforcement while the last nonvariable component is the work of the external forces. The necessary conditions for stationarity of the Lagrange functional J for a fixed outline L and nonvariable stiffness $G(S)$ are the equilibrium differential equations in the domain and the static boundary conditions.

The problem to determine the shape of the hole outline L in the plate and the law of stiffness variations $G(s)$ of the reinforcing rod can be formulated as a variational problem on the stationary value of the functional (1.3) in domains with moving boundaries under the additional conditions (1.2). To solve the isoperimetric variational problem, we form the functional

$$U = J + \lambda_1 \iint_{\Omega'} dx\,dy + \lambda_2 \int_L G(s)\,ds \tag{1.4}$$

where λ_1, λ_2 are constant Lagrange multipliers. For the first variation of the functional (1.4) for a mobile outline L and variable stiffness $G(s)$, we obtain

$$\delta U = -K\iint_{\Omega} [(u_{xx} + \gamma_1 u_{yy} + \gamma_2 v_{xy})\,\delta u_1 + (v_{yy} + \gamma_1 v_{xx} + \gamma_2 u_{xy})\,\delta v_1]\,dx\,dy - K\int_L [T_1(s)\,\delta u_1 + T_2(s)\,\delta v_1]\,ds - \tag{1.5}$$

$$\frac{K}{2}\int_L [u_x^2 + v_y^2 + 2\nu u_x v_y + \gamma_1 (u_y + v_x)^2]\,\delta n\,ds + K\int_{L_1} [T_1(s)\,\delta u + T_2(s)\,\delta v]\,ds + \int_L \Big\{\Big[\frac{1}{2}\varepsilon^2(s) + \lambda_2\Big]\,\delta G_2 +$$

$$\alpha_1'(s)\,(y_n \delta u_2^* - x_n \delta v_2^*) + \frac{1}{\rho}\alpha_1(s)\,(x_n \delta u_2^* + y_n \delta v_2^*) + \Big[\lambda_1 + \frac{1}{\rho}\lambda_2 G(s) - \frac{1}{2\rho}\alpha_2(s) - \beta'(s)\Big]\,\delta n\Big\}\,ds -$$

$$h\int_{L_1} (p x_n \delta u + q y_n \delta v)\,ds$$

$$\delta u_1 = \delta u - u_x \delta x - u_y \sigma y, \quad \delta v_1 = \delta v - v_x \delta x = v_y \delta y$$
$$\delta u_2^* = \delta u^* - u_s^* \delta t, \quad \delta v_2^* = \delta v^* - v_s^* \delta t, \quad \delta G_2 = \delta G - G_s \delta t$$
$$T_1(s) = (u_x + \nu v_y)\,x_n + \gamma_1 (u_y + v_x)\,y_n$$
$$T_2(s) = (v_y + \nu u_x)\,y_n + \gamma_1 (u_y + v_x)\,x_n$$
$$\alpha_1(s) = \varepsilon(s)\,G(s), \quad \alpha_2(s) = \varepsilon^2(s)\,G(s)$$
$$\beta(s) = (u_s^* x_n + v_s^* y_n)\,\alpha_1(s), \quad \gamma_2 = (1 + \nu)/2$$

Here δu, δv, δu^*, δv^*, δG are the total variations of the appropriate quantities, δx, δy are variations of coordinates of points of the outline referred to the $x0y$ coordinate system, δn, δt are variations of points of the outline in a coordinate system associated with the normal n and the tangent t to the required outline L, ρ is the radius of curavture of the outline, and $\varepsilon(s) = -u_s^* y_n + v_s^* x_n$ is the strain of the reinforcing rod.

Let us examine obtaining the variation (1.5) in greater detail. Variation of double integrals with variable domain of integration was examined in a numer of papers (see the appropriate references in /12/, for instance). Let us clarify the evaluation of the variations of the contour integrals in (1.4) with variable outline L:

$$J_1 = \int_L G(s)\,ds, \quad J_2 = \frac{1}{2}\int_L G(s)\,(-u_s^* y_n + v_s^* x_n)^2\,ds$$

A method is described in /13/ to obtain variations of a multiple integral with moving boundaries for which the independent variables are considered as functions of other auxiliary nonvariable variables with fixed range of variation. Extending the same method to the calculation of the variations of the contour integrals J_1, J_2, we assume the arclength coordinate s and all the qantities in the integrand are functions of a certain nonvariable parameter ξ

$$s = s(\xi), \quad ds = (x_\xi^2 + y_\xi^2)^{1/2} d\xi, \quad J_1 = \int_{L^*} G[s(\xi)]\,(x_\xi^2 + y_\xi^2)^{1/2} d\xi, \quad J_2 = \frac{1}{2}\int_{L^*} G[s(\xi)]\left(\frac{u_\xi^* x_\xi + v_\xi^* y_\xi}{x_\xi^2 + y_\xi^2}\right)^2 (x_\xi^2 + y_\xi^2)^{1/2} d\xi$$

where L^* is a certain fixed outline. We consequently arrive at the usual problem of calculating the contour integrals with a fixed contour. For δJ_1 we have

$$\delta J_1 = \int_{L^*}\left[\delta G\,(x_\xi^2 + y_\xi^2)^{1/2} + G\,\frac{x_\xi \delta x_\xi + y_\xi \delta y_\xi}{(x_\xi^2 + y_\xi^2)^{1/2}}\right] d\xi = \int_{L^*}\left[\delta G\,(x_\xi^2 + y_\xi^2)^{1/2} - \frac{dG}{d\xi}\,\frac{x_\xi \delta x + y_\xi \delta y}{(x_\xi^2 + y_\xi^2)^{1/2}} + \frac{G}{\rho}(y_\xi \delta x - x_\xi \delta y)\right] d\xi$$

$$\frac{1}{\rho} = \frac{x_\xi y_{\xi\xi} - x_{\xi\xi} y_\xi}{(x_\xi^2 + y_\xi^2)^{3/2}}$$

Returning to the initial variable s and taking account of the equalities

$$\delta n = x_n \delta x + y_n \delta y, \quad \delta t = x_s \delta x + y_s \delta y, \quad x_n = y_s, \quad y_n = -x_s$$

and the notation in (1.5), we obtain

$$\delta J_1 = \int_L \left(\delta G_2 + \frac{G}{\rho} \delta n \right) ds$$

We perform analogous actions in evaluating δJ_2

$$\delta J_2 = \frac{1}{2} \int_{L^*} \left[\delta G \frac{(u_\xi{}^* x_\xi + v_\xi{}^* y_\xi)^2}{(x_\xi{}^2 + y_\xi{}^2)^{3/2}} + 2G \frac{u_\xi{}^* x_\xi + v_\xi{}^* y_\xi}{(x_\xi{}^2 + y_\xi{}^2)^{3/2}} (x_\xi \delta u_\xi{}^* + y_\xi \delta v_\xi{}^* + u_\xi{}^* \delta x_\xi + v_\xi{}^* \delta y_\xi) - 3G \frac{(u_\xi{}^* x_\xi + v_\xi{}^* y_\xi)^2}{(x_\xi{}^2 + y_\xi{}^2)^{5/2}} (x_\xi \delta x_\xi + y_\xi \delta y_\xi) \right] d\xi$$

Integrating by parts, going over to the variable s and taking into account that

$$\delta x = x_s \delta t + y_s \delta n, \quad \delta y = y_s \delta t - x_s \delta n, \quad x_{ss} = -y_s/\rho, \quad y_{ss} = x_s/\rho$$

we find

$$\delta J_2 - \int_L \left\{ \frac{1}{2} \varepsilon^2 \delta G_2 + \alpha_1'(s) (y_n \delta u_2{}^* - x_n \delta v_2{}^*) + \frac{1}{\rho} \alpha_1(s) (x_n \delta u_2{}^* + y_n \delta v_2{}^*) - \left[\frac{1}{2\rho} \alpha_2(s) + \beta'(s) \right] \delta n \right\} ds$$

where the notation agrees with that taken in (1.5).

Equilibrium differential equations in the domain Ω and static conditions on a sufficiently remote contour L_1 for the plate correspond to the stationarity condition of the functional (1.4). Taking account of the continuity of the displacements $u = u^*$, $v = v^*$ on the boundary L, we obtain equailibrium equations for a rod element interacting with the adjacent plate

$$-h\sigma_n + \alpha_1(s) / \rho = 0, \quad h\tau_{tn} + \alpha_1'(s) = 0 \tag{1.6}$$

because of the arbitrariness of the variations $\delta u, \delta v$ $(\delta u^*, \delta v^*)$.

Here σ_n, τ_{tn} are the normal and tangential components of the stress vector acting on an area with normal n. Because of the variation of the reinforcement stiffness $G(s)$, we have a condition on L

$${}^1/_2 \varepsilon^2(s) + \lambda_2 = 0 \tag{1.7}$$

from which the constancy of the strain (equal intensity) of the reinforcing rod follows

$$\varepsilon(s) = -u_s{}^* y_n + v_s{}^* x_n = C_1 \quad (C_1 = \text{const}) \tag{1.8}$$

We find the condition on the hole outline

$$K \{T_1(s) u_n + T_2(s) v_n - {}^1/_2 [u_x{}^2 + v_y{}^2 + 2\nu u_x v_y + \gamma_1 (u_y + v_x)^2]\} - \rho^{-1} \alpha_2(s) - \beta'(s) + \lambda_1 = 0 \tag{1.9}$$

because of the arbitrariness of the variation δn. The relationship (1.7) is taken into account here.

We go over to the stresses $\sigma_n, \sigma_t, \tau_{tn}$ in (1.9), where σ_t is the normal component of the stress vector acting on an area with normal t coincident with the tangent direction to the outline. To do this we use (1.6), the notation (1.5), the plate equilibrium differential equations, the continuity conditions for the displacements on L, the generalized Hooke's law, and the Cauchy eleasticity relationships. In differentiating the directional cosines x_n, y_n we take into account that

$$x_{ns} = -y_n / \rho, \; y_{ns} = x_n / \rho, \; x_{nn} = y_{nn} = 0$$

i.e., that in contrast to the derivatives with respect to the arclength coordinate s, the derivatives with respect to the normal along the contour are zero just as along any other straight line. Consequently, condition (1.9) on L takes the form

$$-{}^1/_2 (\sigma_n + \sigma_t)^2 + (1 + \nu)(\tau_{tn}{}^2 - \sigma_n \sigma_t) + 2(1 + \nu)\sigma_n{}^2 + \sigma_n \rho [(2 + \nu) \partial\sigma_n / \partial n + \partial\sigma_t / \partial n] = C_2 \; (C_2 = \text{const}) \tag{1.10}$$

As in /1/, we obtain the additional conditions (1.8) and (1.10) on L as natural conditions of the variational problem on the stationary value of the functional U for a movable contour L and a variatible stiffness $G(s)$ in addition to the usual equations and boundary conditions for the problem under consideration. These conditions permit determination of the law of stiffness under the tension $G(s)$ of the reinforcement and the shape of the contour L.

Remark 1. If the problem of determining the shape of an unreinforced hole in a plate is examined under the same loading conditions (1.1) from the condition of minimum elastic strain energy for a given area of a hole enclosed by a contour, the condition of constancy of the stress intensity is obtained as an additional condition on the contour. This result can be obtained from the stationarity condition in the domain Ω with moving boundary L for the functional (1.4) in which the components associated with reinforcement must be omitted. The same condition was obtained in /14/, where the minimum of the maximum value of stress intensity in the region occupied by the plate with attached boundaries was taken as the criterion for the optimum hole shape. It is shown in /15/ that an equal-stressed hole outline is optimal from the viewpoint of stress minimization in a plate. The problem of determining the shape of an equally strong outline of an unreinforced hole in a plate stretched in two directions is solved in /16/.

Remark 2. The integral criterion to determine the reinforcement of a hole in another shape was proposed for a shell in /17/, and in application to the reinforcement of a circular hole in a plate in /18/. The stiffness characteristics of the reinforcing rod, which are constant along a hole outline of given shape, were sought from the condition of minimum energy of the additional state of stress (due to the presence of the reinforced hole) of the shell (plate).

2. Transformation of the equations by using a complex representation of the stresses and displacements. Solution for the case of equilateral tension on a plate. Let the exterior of a unit circle $|\zeta| \geqslant 1$ in the ζ plane be mapped onto the exterior of the desired contour L in the plane $z = x + iy$ by means of the function

$$\frac{z}{B} = \omega(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^{1-2k} \quad (b_0 = 1) \tag{2.1}$$

where B is a real quantity defining the scale, and b_k are unknown real coefficients. Let the complex variable ζ on the unit circle henceforth be denoted by $\tau = \exp(i\theta), 0 \leqslant \theta < 2\pi$.

Let $\Phi(\zeta)$, $\Psi(\zeta)$ be the Kolosov-Muskhelishvili functions in the transformed domain $|\zeta| \geqslant 1$ that describe the plane state of stress in the plate. In conformity with /19/, and taking account of symmetry, these functions have the form

$$\Phi(\zeta) = \Gamma + \sum_{k=1}^{\infty} R_k \zeta^{-2k}, \quad \Psi(\zeta) = \Gamma' + \sum_{k=1}^{\infty} Q_k \zeta^{-2k} \tag{2.2}$$

Here R_k, Q_k are certain real coefficients, the constants Γ, Γ' are determined from the conditions at infinity (1.1)

$$\Gamma = (p + q) / 4, \quad \Gamma' = -(p - q) / 2 \tag{2.3}$$

We seek the unknown function $G(s)$ (the stiffness of the reinforcing rod under tension) in the form of a Fourier series expansion

$$\frac{G}{A} = \sum_{k=0}^{\infty} F_k \cos(2k\theta) \quad (F_0 = 1) \tag{2.4}$$

where F_k are unknown coefficients. Henceforth, we use a function of the complex variable τ (a point of the unit circle)

$$g(\tau) = \sum_{k=0}^{\infty} F_k \tau^{-2k} \quad (F_0 = 1) \tag{2.5}$$

in place of (2.4), such that $G / A = \operatorname{Re} g(\tau)$.

Using the functions (2.1)–(2.3) and (2.5) introduced, the boundary conditions (1.6), (1.8), (1.10) are converted, respectively, to the form

$$\begin{gathered}
-4(1+\nu)^{-1}\operatorname{Re}\Phi(\tau) + \mu\,[(1+\nu)^{-1} + \lambda N |\omega'(\tau)|^{-3} \operatorname{Re} g(\tau)] = 0 \\
\operatorname{Im}\,[P(\tau) + \lambda\mu\tau g'(\tau) |\omega'(\tau)|^{-1}] = 0 \\
\operatorname{Re}\,[2(1-\nu)\Phi(\tau) - (1+\nu)P(\tau)] - \mu = 0 \quad (\mu = \text{const}) \\
-(1+\nu)^{-1}\{4M \operatorname{Re}\Phi(\tau) + \mu\,[8(3-\nu)(1+\nu)^{-1}\operatorname{Re}\Phi(\tau) - \\
M]\} - \operatorname{Re}\,[P^2(\tau)] + 4(1-\nu)(7-\nu)(1+\nu)^{-2}[\operatorname{Re}\Phi(\tau)]^2 - \\
C = 0 \quad (C = \text{const})
\end{gathered} \tag{2.6}$$

Here

$$\begin{gathered}
N = |\omega'(\tau)|^2 + \operatorname{Re}\,[\tau\omega''(\tau)\overline{\omega'(\tau)}] \\
P(\tau) = -\tau^2\,[\overline{\omega(\tau)}\,\Phi'(\tau) + \omega'(\tau)\Psi(\tau)] / \overline{\omega'(\tau)} \\
M = N^{-1}\operatorname{Re}\{-(5+\nu)(1+\nu)^{-1}\tau\,|\omega'(\tau)|^2\,\Phi'(\tau) + \\
\tau^3\,[\overline{\omega(\tau)}\,(\Phi''(\tau)\,\omega'(\tau) - \Phi'(\tau)\,\omega''(\tau)) + (\omega'(\tau))^2\,\Psi'(\tau)]\} \\
\mu = 2E\varepsilon / (p + q), \quad \lambda = A / (EhB)
\end{gathered} \tag{2.7}$$

Four functional equations (2.6) have been obtained to determine the functions

$$\Phi(\tau)=\frac{1}{2}+\sum_{k=1}^{\infty}R_k\tau^{-2k},\quad \Psi(\tau)=-V+\sum_{k=1}^{\infty}Q_k\tau^{-2k},\ \omega(\tau)=\sum_{k=0}^{\infty}b_k\tau^{1-2k},\quad g(\tau)=\sum_{k=0}^{\infty}F_k\tau^{-2k}\quad (b_0=F_0=1) \tag{2.8}$$

where $\Phi(\tau)$, $\Psi(\tau)$ are the boundary values of the appropriate dimensionless functions $\Phi(\zeta)$, $\Psi(\zeta)$ obtained from (2.2) and (2.3) by dividing them by $(p+q)/2$. The coefficients R_k, Q_k, b_k, F_k in (2.8), the constant C in the last condition in (2.6), and the parameter μ related to the strain ε of the reinforcement by the notation (2.7) are unkown. The relative stiffness parameters under tension λ and the nonsymmetry of the load

$$V=(p-q)/(p+q) \tag{2.9}$$

are considered given.

Let us note that the first, second, and fourth conditions in (2.6) correspond to (1.6), (1.10) only in taking account of the third condition in (2.6) of the constancy of the strain of the reinforcing rod.

In the particular case when the stresses in the plate are identical at infinity $(p=q)$, the solution of the problem under consideration is a circular hole reinforced by a constant-stiffness elastic rod. For $V=0,\ b_k=F_k=0\ (k\geqslant 1)$, we have from the first and second conditions in (2.6)

$$R_k=0\ (k\geqslant 1),\quad Q_k=0\ (k\geqslant 2),\quad Q_1=\frac{1-\lambda(1-\nu)}{1+\lambda(1+\nu)} \tag{2.10}$$

We obtain values of the constants μ, C from the third and fourth conditions in (2.6)

$$\mu=\frac{2}{1+\lambda(1+\nu)},\quad C=-\left(\frac{5-3\nu}{1+\nu}\left(2Q_1+\frac{1-\nu}{1+\nu}\right)-3Q_1^2\right) \tag{2.11}$$

In the equivalent reinforcement problem /1/, the solution in the equilateral tension case is a circular hole reinforced by a constant-stiffness elastic rod. However, the value of the parameter λ is defined strictly and equals $(1-\nu)^{-1}$. The parameter $\lambda=A/(EhB)$ is given in the problem under consideration.

3. Construction of the approximate solution by the small parameter method. In solving the inverse boundary value problems of elasticity theory when the domain boundary determined from certain additional conditions in unknown, the small parameter method /20,21/ turns out to be sufficiently effective.

Let the stress (1.1) act in the directions of the $0x, 0y$ axes in a plate at infinity, where the load nonsymmetry parameter (2.9) for V is considered small. We seek the coefficients R_k, Q_k, b_k, F_k in (2.8), and the constants μ, C in conditions (2.6) in the form of expansions in powers of V

$$R_k=\sum_{j=k}^{\infty}R_k^{(2j-k)}V^{2j-k},\quad Q_k=\sum_{j=k}^{\infty}Q_k^{(2j-k-1)}V^{2j-k-1},\quad b_k=\sum_{j=k}^{\infty}b_k^{(2j-k)}V^{2j-k},\quad F_k=\sum_{j=k}^{\infty}F_k^{(2j-k)}V^{2j-k}\quad (k=1,2,3,\ldots) \tag{3.1}$$

$$\mu=\sum_{j=0}^{\infty}\mu^{(2j)}V^{2j},\quad C=\sum_{j=0}^{\infty}C^{(2j)}V^{2j}$$

For $V=0$, i.e., for $p=q$, only coefficients of the zero approximation remain, $Q_1=Q_1^{(0)}$, $\mu=\mu^{(0)}, C=C^{(0)}$, that agree with the coresponding quantities in (2.10), (2.11).

Substituting (2.8), (3.1) into the boundary conditions (2.6), we obtain a system of equations, for identical powers of τ, for the quantities of the zero approximation for V^0, the first for V^1, the second for V^2, etc. Since the coefficients of the preceding approximations are calculated, the system of equations to determine the coefficients of the next approximation turns out to be linear.

Coefficients of the zero, first, and second approximations were calculated for the actual solution. The computation results are presented in Figs.2—7. The Poisson ratio ν was taken equal to 0.3.

Hole contours L are represented in Fig.2 for $\lambda=A/(EhB)=0.3$ and also the corresponding dimensionless stiffnesses under tension G/A of the reinforcing rod as a function of the angle θ. Displayed in Fig.3 are the contours L and stiffnesses G/A of the reinforcement for $V=(p-q)/(p+q)=0.2$ and different values of λ. The distribution of the stress concentration factor α in the plate is shown in Fig.4 along the contour L of the cutout (along the angle θ) for $\lambda=0.5$ (dashed lines), $\lambda=1.0$ (solid lines) for a number of values of the load parameter V.

The stress concentration factor α is calculated as the ratio of the stress intensity at a given point to the stress intensity at infinity

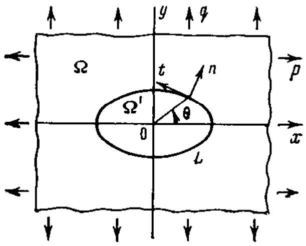

Fig.1

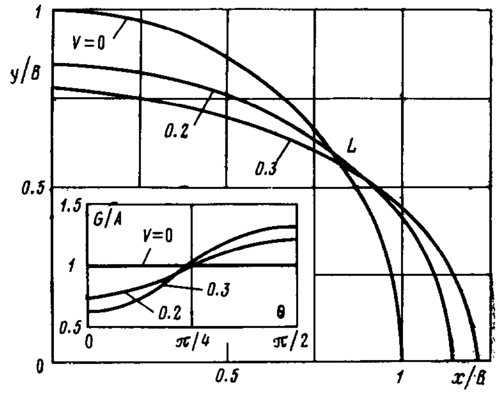

Fig.2

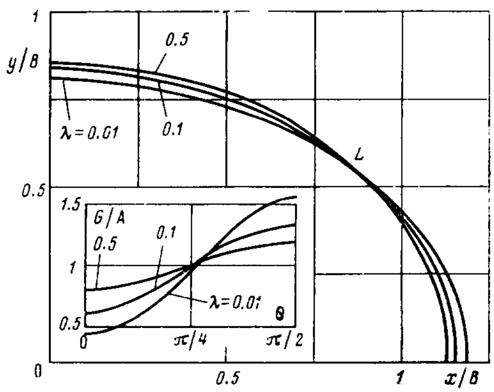

Fig.3

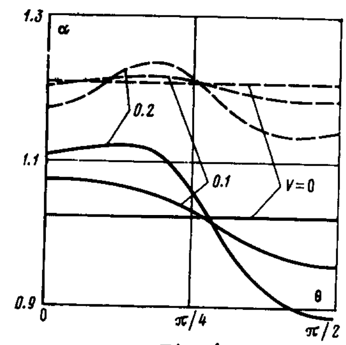

Fig.4

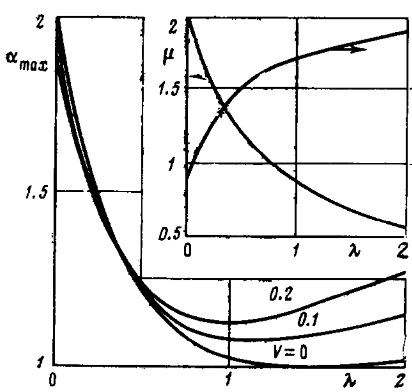

Fig.5

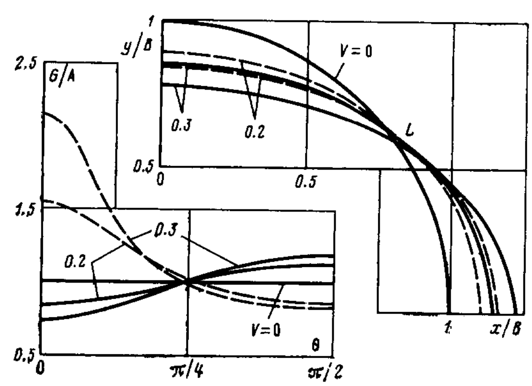

Fig.6

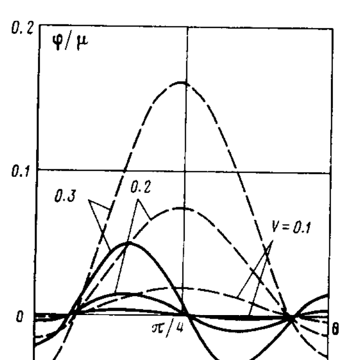

Fig.7

$$\alpha = \sigma_i / \sigma_{i\infty}, \ \sigma_i = (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2)^{1/2}, \ \sigma_{i\infty} = (p^2 + q^2 - pq)^{1/2} \quad (3.2)$$

Dependences of the maximum value of the stress concentration factor α_{max} on the cutout outline in a plate on the relative stiffness parameter λ are represented in Fig.5 for values of V indicated on the curves. The curves have a minimum, where as the load nonsymmetry parameter V increases the value of the least possible value of α_{max} grows. Dependences of the parameter $\mu = 2E\varepsilon / (p + q)$, related to the strain of the reinforcing rod, and of the constant C in the last condition of (2.6) on λ are also presented for $V = 0.2$. The change in μ, C versus V in the range $0 \leqslant V \leqslant 0.3$ is insubstantial.

It is of definite interest to compare the solution obtained to the solution of the problem of determining the equivalent reinforcement /1/, also selected in the form of a momentless rod. We take the parameter λ equal to $(1 - \nu)^{-1} \approx 1.429$ (for $\nu = 0.3$), and both solution will agree for $V = 0$.

Hole outlines in the problem of equivalent reinforcement (dashed lines) and the holes found (solid lines) are displayed in Fig.6 for a number of values of V. Also presented for comparison are the stiffnesses under tension of the equivalent reinforcement and the reinforcement obtained from the condition of minimum energy of a plate with a reinforcement.

The greatest stiffness for the reinforcement found corresponds to intersections of the hole outline with the axis of symmetry $0y$ along the direction of least force action ($\theta = \pi / 2$, $\theta = 3\pi / 2$), and the least stiffness to the points $\theta = 0, \theta = \pi$. A qualitatively different picture is observed for the equivalent reinforcement: the greatest stiffness at the points $\theta = 0, \theta = \pi$ and the least at the points $\theta = \pi / 2, \theta = 3\pi / 2$. The equivalent reinforcement works under higher stressed conditions. For instance, for $V = 0.1$ the maximal stress in the equivalent reinforcement, referred to the stress intensity in a plate at infinity, holds at the points $\theta = \pi/2$, $\theta = 3\pi/2$, and equals 0.818. An analogous stress in the found reinforcement, which is constant over the outline, equals 0.690. But by comparison with the problem of the equivalent reinforcement, the greatest value of the stress concentration factor (3.2) in a plate on the cutout outline is greater than 1 and equals 1.091.

As computations showed, the hole outlines and the stiffness curves under tension practically agree for the first and second approximation solutions up to load parameter values V (small parameter) equal to 0.3. However, the second approximation improves the accuracy of satisfying conditions (2.6) substantially.

Presented in Fig.7 for $\lambda = 0.3$ is the change in the quantity φ / μ as a function of the angle θ, where $\varphi(\theta)$ denotes the left side of the third condition in (2.6). The dashed lines correspond to the first approximation, and the solid lines to the second.

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REFERENCES

1. MANSFIELD E.H., Neutral holes in plane sheet: reinforced holes which are elastically equivalent of the uncut sheet. Aeronaut. Res. Council Repts. and Mem. No.2815, 1950.

2. SHEREMET'EV M.P., Plate state of stress of a plate with reinforced circular hole. Inzh. Sb., Vol.14, 1953.

3. SAVIN G.N. and TUL'CHII V.I., Plates Reinforced by Composite Rings and Elastic Cover-plate. "Naukova Dumka", Kiev, 1971.

4. RADOK J.R.M., Problems of plane elasticity for reinforced boundaries. Trans. ASME, J. Appl. Mech., Vol.22, No.2, 1955.

5. WELLS A.A., On the plane stress-distribution in an infinite plate with a rim-stiffened elliptical opening. Quart. J. Mech. and Appl. Math., Vol.3, Pt. 1, 1950.

6. MANSFIELD E.H., Optimum design for reinforced circular holes. Aeronaut. Res. Council Current Papers No. 239, 1956.

7. MANSFIELD E.H. and HANSON C.J., Optimum reinforcement around a circular hole in a flat sheet under uniaxial tension. Aeronaut.Res.Council Repts. and Mem.No.3723,1973.

8. HICKS R., Reinforced elliptical holes in stressed plates. J. Roy. Aeronaut. Soc., Vol.61, No. 562, 1957.

9. HICKS R., Variably reinforced circular holes in stressed plates. Aeronaut. Quart., Vol.9, Pt. 3, 1958.

10. SAVIN G.N., and FLEISHMAN N.P., Plates Whose Edges are Reinforced by Thin Ribs, Prikl. Mekhan., Vol.7, No.4, 1961.

11. TUL'CHII V.I., On the optimal reinforcement of holes in plates, Prikl. Mekhan., Vol.1, No.3, 1965.

12. KURSHIN L.M., On the problem of determining the shape of a rod section by the maximal torsional stiffness, Dokl. Akad. Nauk SSSR, Vol.223, No.3, 1975.

13. BOLZA O., Vorlesungen über Variationsrechnung. Kochler, Leipzig, 1949.

14. BANICHUK N.V., Optimality conditions in the problem of seeking the hole shape in elastic bodies. PMM, Vol.41, No.5, 1977.

15. WHEELER L., On the role of constant-stress surface in the problem of minimizing elastic stress concentration, Intern. J. Solids and Struct., Vol.12, No.11, 1976.

16. CHEREPANOV G.P., Some problems of elasticity and plasticity theory with unknown boundary. IN: Application of Function Theory in the Mechanics of a Continuous Medium. Vol.1, "Nauka", Moscow, 1965.

17. MIKHAILOVSKII E.I., On the optimal reinforcement of shell edge. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1975.

18. MIKHAILOVSKII E.I.,and CHAUNIN M.P., Rational reinforcement of a circular hole in a plane plate being stretched, Problemy Prochnosci,No.1,1978.

19. MUSKHELISHVILI N.I., Some Fundamental Problems of Mathematical Elasticity Theory, "Nauka", Mosow, 1966.

20. ALEKSANDROV A.Ia., GORBATYI A.V., and KURSHIN L.M., On the solution of the problem of the equivalent hole reinforcement, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.4, 1969.

21. BANICHUK N.V., On a variational problem with unknown boundaries and the determination of optimal shapes of elastic solids, PMM Vol.39, No.6, 1975.

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